

Conical Singularities in Three or Four - Dimensions and Supersymmetry Breaking¹

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Abstract

We present mechanisms for generating conical singularities both in three and four-dimensions in the systems with copies of scalar or chiral multiplets coupled to $N = 2$ or $N = 1$ supergravity. Our mechanisms are useful for supersymmetry breaking, maintaining the zero cosmological constants in three and four-dimensions. A strong coupling duality connecting these two dimensionalities is also studied.

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1. Introduction

It is some years ago that E. Witten proposed an original scenario of supersymmetry breaking [1] maintaining the zero cosmological constant in three-dimensional space-time ($D = 3$) with conical singularities [2]. Such a conical space-time has a deficit angle which lifts the degeneracy between bosons and fermions in a supermultiplet yielding supersymmetry breaking. An explicit mechanism realizing this mechanism has also been presented by K. Becker *et al.* [3] based on $D = 3$, $N = 2$ supergravity coupled to a vector multiplet and a charged scalar multiplet, with a Nielsen-Olesen vortex soliton solution [4]. However, there has been no indication that a direct analog of this mechanism works in four-dimensions [3].

Recently there has been a nice scenario that the mystery about the vanishing cosmological constant in $D = 4$ even after the supersymmetry breaking can be solved by the strong coupling duality [5] between the $D = 3$ theory with broken supersymmetry with conical singularity and the $D = 4$ theory regarded as its strong coupling limit $\lambda \rightarrow \infty$. In particular, the $D = 4$ Poincaré invariance with a zero cosmological constant is recovered, because the radius r of the circle in $\mathbb{R}^3 \otimes S^1$ grows: $r \rightarrow \infty$ as $\lambda \rightarrow \infty$ [5].

In this paper we present an explicit model of generating such conical singularity for $D = 3$, $N = 2$ supergravity different from that in [3], by taking a limit of a particular mass to infinity, which is applicable also to $D = 4$, $N = 1$ supergravity resulting in a desirable supersymmetry breaking maintaining the vanishing cosmological constant. We also see if the recent scenario of the strong coupling duality [5] works between the two systems we deal with.

2. $D = 3$, $N = 2$ Supergravity Coupled to Scalar Multiplets

Before presenting our mechanism of conical singularity, we first fix the invariant lagrangian of the $D = 3$, $N = 2$ supergravity. Our field contents are the supergravity multiplet $(e_\mu^m, \psi_\mu^i, B_\mu, \chi^i, \varphi)$ with the indices $i, j, \dots = 1, 2$ for the doublets under the global $SO(2)$ of the $N = 2$ supersymmetry, and n copies of scalar multiplets (A_a, B_a, λ_a^i) ($a, b, \dots = 1, \dots, n$). The indices $\mu, \nu, \dots = 0, 1, 2$ are for curved coordinates, while $m, n, \dots = (0), (1), (2)$ are local Lorentz indices with the signature $(+, -, -)$. We do not gauge the global $SO(2)$ symmetry in order not to create a potential with a generally non-zero cosmological constant.

Our invariant lagrangian for $N = 2$ supergravity is³

$$\begin{aligned}
e^{-1} \mathcal{L}_{\text{SG}} = & -\frac{1}{4} R + \frac{1}{2} e^{-1} \epsilon^{\mu\nu\rho} (\bar{\psi}_\mu{}^i D_\nu(\hat{\omega}) \psi_\rho{}^i) - \frac{1}{4} e^{2b\varphi} G_{\mu\nu}^2 + \frac{i}{2} (\bar{\chi}^i \gamma^\mu D_\mu(\hat{\omega}) \chi^i) \\
& + \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{i}{2\sqrt{2}} \epsilon^{ij} (\bar{\psi}_\mu{}^i \gamma^{\rho\sigma} \gamma^\mu \chi^j) e^{b\varphi} G_{\rho\sigma} - \frac{1}{\sqrt{2}} (\bar{\psi}_\mu{}^i \gamma^\nu \gamma^\mu \chi^i) \partial_\nu \varphi \\
& - \frac{1}{2b} \epsilon^{ij} e^{b\varphi} (\bar{\psi}_\mu{}^i \psi_\nu{}^j) G^{\mu\nu} + \frac{1}{4} \left(\frac{1}{b} - b \right) \epsilon^{ij} e^{b\varphi} (\bar{\chi}^i \gamma^{\mu\nu} \chi^j) G_{\mu\nu} \quad .
\end{aligned} \tag{2.1}$$

³To our knowledge, the lagrangians (2.1) - (2.3) have never been presented in the past. These lagrangians are fixed up to fermionic quartic terms.

The constant b is an arbitrary non-zero real number. The kinetic lagrangian for the scalar multiplets is

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{SM}} = \sum_a \Big[& +\frac{1}{2}e^{-b\varphi}(\partial_\mu A_a)^2 + \frac{1}{2}e^{-b\varphi}(\partial_\mu B_a)^2 + \frac{i}{2}(\bar{\lambda}_a^i \gamma^\mu D_\mu(\widehat{\omega})\lambda_a^i) \\
& -\frac{1}{\sqrt{2}}(\bar{\psi}_\mu^i \gamma^\nu \gamma^\mu \lambda_a^i) e^{-b\varphi/2} \partial_\nu A_a - \frac{1}{\sqrt{2}}\epsilon^{ij}(\bar{\psi}_\mu^i \gamma^\nu \gamma^\mu \lambda_a^j) e^{-b\varphi/2} \partial_\nu B_a \\
& +be^{-1}\epsilon^{\mu\nu\rho}B_\mu(\partial_\nu A_a)(\partial_\rho B_a) - \frac{1}{8}\left(b - \frac{2}{b}\right)\epsilon^{ij}(\bar{\lambda}_a^i \gamma^{\mu\nu}\lambda_a^j) e^{b\varphi}G_{\mu\nu} \\
& +\frac{i}{2}b(\bar{\chi}^i \gamma^\mu \lambda_a^i) e^{-b\varphi/2} \partial_\mu A_a + \frac{i}{2}b\epsilon^{ij}(\bar{\chi}^i \gamma^\mu \lambda_a^j) e^{-b\varphi/2} \partial_\mu B_a \Big] .
\end{aligned} \tag{2.2}$$

The invariant mass terms are⁴

$$\begin{aligned}
e^{-1}\mathcal{L}_m = \sum_a \Big[& -\frac{1}{2}m_a^2 e^{-b\varphi}(A_a^2 + B_a^2) + \frac{1}{2}m_a(\bar{\lambda}_a^i \lambda_a^i) \\
& +\frac{i}{\sqrt{2}}m_a(\bar{\psi}_\mu^i \gamma^\mu \lambda_a^i) e^{-b\varphi/2} A_a + \frac{i}{\sqrt{2}}m_a\epsilon^{ij}(\bar{\psi}_\mu^i \gamma^\mu \lambda_a^j) e^{-b\varphi/2} B_a \\
& -\frac{1}{2}bm_a(\bar{\chi}^i \lambda_a^i) e^{-b\varphi/2} A_a - \frac{1}{2}bm_a\epsilon^{ij}(\bar{\chi}^i \lambda_a^j) e^{-b\varphi/2} B_a \Big] .
\end{aligned} \tag{2.3}$$

The total lagrangian $\mathcal{L}_{\text{total}} \equiv \mathcal{L}_{\text{SG}} + \mathcal{L}_{\text{SM}} + \mathcal{L}_m$ up to quartic fermion terms is invariant under the supertranslation rules up to bilinear fermion terms

$$\delta e_\mu{}^m = -i(\bar{\epsilon}^i \gamma^m \psi_\mu^i) , \tag{2.4a}$$

$$\delta \psi_\mu^i = D_\mu(\widehat{\omega})\epsilon^i + \frac{1}{2b}e^{-1}\epsilon_\mu{}^{\rho\sigma}\epsilon^{ij}\epsilon^j e^{b\varphi}\widehat{G}_{\rho\sigma} , \tag{2.4b}$$

$$\delta B_\mu = \frac{i}{\sqrt{2}}e^{-b\varphi}\epsilon^{ij}(\bar{\epsilon}^i \gamma_\mu \chi^j) - \frac{1}{b}e^{-b\varphi}\epsilon^{ij}(\bar{\epsilon}^i \psi_\mu^j) , \tag{2.4c}$$

$$\delta \chi^i = -\frac{1}{2\sqrt{2}}\epsilon^{ij}\gamma^{\mu\nu}\epsilon^j e^{b\varphi}\widehat{G}_{\mu\nu} - \frac{i}{\sqrt{2}}\gamma^\mu \epsilon^i \widehat{D}_\mu \varphi , \tag{2.4d}$$

$$\delta \varphi = +\frac{1}{\sqrt{2}}(\bar{\epsilon}^i \chi^i) , \tag{2.4e}$$

$$\delta A_a = \frac{1}{\sqrt{2}}e^{b\varphi/2}(\bar{\epsilon}^i \lambda_a^i) , \tag{2.5a}$$

$$\delta B_a = \frac{1}{\sqrt{2}}\epsilon^{ij}e^{b\varphi/2}(\bar{\epsilon}^i \lambda_a^j) , \tag{2.5b}$$

$$\begin{aligned}
\delta \lambda_a^i = & -\frac{i}{\sqrt{2}}(\gamma^\mu \epsilon^i) e^{-b\varphi/2} \partial_\mu A_a + \frac{i}{\sqrt{2}}\epsilon^{ij}(\gamma^\mu \epsilon^j) e^{-b\varphi/2} \partial_\mu B_a \\
& +\frac{1}{\sqrt{2}}m_a\epsilon^i e^{-b\varphi/2} A_a - \frac{1}{\sqrt{2}}m_a\epsilon^{ij}\epsilon^j e^{-b\varphi/2} B_a .
\end{aligned} \tag{2.5c}$$

As usual [6], this system has a global symmetry associated with the dilaton:

$$\varphi \rightarrow \varphi + c , \quad B_\mu \rightarrow e^{-bc}B_\mu , \quad A_a \rightarrow e^{bc/2}A_a , \quad B_a \rightarrow e^{bc/2}B_a , \tag{2.6}$$

with an arbitrary non-zero real constant parameter c . Additionally, these lagrangians are scaling:

$$\mathcal{L}_{\text{SG}} \rightarrow e^{ac}\mathcal{L}_{\text{SG}} , \quad \mathcal{L}_{\text{SM}} \rightarrow e^{ac}\mathcal{L}_{\text{SM}} , \quad \mathcal{L}_m \rightarrow e^{ac}\mathcal{L}_m , \tag{2.7a}$$

⁴We have not been successful for finding out a more general potential terms.

under

$$\begin{aligned}
\varphi &\rightarrow \varphi + c \quad , \\
(e_\mu{}^m, \psi_\mu, B_\mu, \chi) &\rightarrow (e^{ac} e_\mu{}^m, e^{ac/2} \psi_\mu, e^{(a-b)c} B_\mu, e^{-ac/2} \chi) \quad , \\
(A_a, B_a, \lambda) &\rightarrow (e^{bc/2} A_a, e^{bc/2} B_a, e^{-ac/2} \lambda) \quad , \quad m \rightarrow e^{-ac} m \quad ,
\end{aligned} \tag{2.7b}$$

where c is an arbitrary constant parameter, while a is a real constant. (For simplicity we can choose $a = b$.)

This system has a zero cosmological constant, unless the global $SO(2)$ symmetry is gauged by minimal couplings with the B_μ -field. We can further introduce a supersymmetric Chern-Simons lagrangian with a $B \wedge G$ -term with a non-zero potential term, which we skip in this paper due to its irrelevance to our present purpose.

There are some remarks in order. The $D = 3, N = 2$ system presented here is related to eqs. (B.1) - (B.4) given in ref. [7] in the way that the irreducible supergravity multiplet $(e_\mu{}^m, \psi_\mu)$ and a vector multiplet (A_μ, λ, S) in [7] are combined to form a reducible supergravity multiplet (2.4) with (A_μ, λ, S) replaced by (B_μ, χ, φ) . In particular, the scalar field S in [7] is now the dilaton in our system with non-polynomial couplings. This can be understood as follows. A simple computation reveals that the kinetic lagrangian for the vector multiplet in [7] generates what is called “improvement term” $\approx S^2 R(\omega)$ in the lagrangian. In fact, the supertranslation of the auxiliary field D for the vector multiplet [7] indicates the gaugino field equation with the term $\approx (\gamma^{\mu\nu} D_{[\mu} \psi_{\nu]}) S$ as in (B.4) in [7], implying the existence of the term $\approx (\bar{\lambda} \gamma^{\mu\nu} D_{[\mu} \psi_{\nu]}) S$ in the lagrangian, which in turn necessitates the improvement term $\approx S^2 R(\omega)$ above. This signals the mixture between the dreibein and the S -field. To have a *canonical* system free from such a mixture, we have to rescale the dreibein in such a way that this scalar appears in the lagrangian with non-polynomial couplings. The simplest principle is to interpret this scalar as the dilaton with exponential couplings, as we have done here. Accordingly, there arise cross terms in the supertranslation rules between the original supergravity and vector multiplets, such as the G -term in (2.4b). It is for this reason that we are dealing with our enlarged supergravity multiplet in this paper.⁵

3. Conical Singularity in $D = 3$

We now present our mechanism generating conical singularities in $D = 3$. We start with the bosonic field equations from $\mathcal{L}_{\text{total}}$:

⁵From these considerations, we do not completely agree with the polynomial couplings of the N -field in ref. [3]. We claim that either non-polynomial couplings of the N -field, or the improvement term $\approx N^2 R(\omega)$ should arise in the lagrangian. This is also reasonable from the viewpoint of duality [5], connecting $D = 4$ and $D = 3$ supergravities, because the dilaton in the former will disappear in the latter under the strong coupling duality. (Cf. (5.3))

$$R_{\mu\nu} = -2e^{2b\varphi}G_{\mu\rho}G_{\nu}{}^{\rho} + g_{\mu\nu}e^{2b\varphi}G_{\rho\sigma}^2 + 2(\partial_{\mu}\varphi)(\partial_{\nu}\varphi) + \sum_a \left[2e^{-b\varphi}(\partial_{\mu}A_a)(\partial_{\nu}A_a) + 2e^{-b\varphi}(\partial_{\mu}B_a)(\partial_{\nu}B_a) - 2m_a^2g_{\mu\nu}e^{-b\varphi}(A_a^2 + B_a^2) \right] , \quad (3.1)$$

$$D_{\mu}^2\varphi = -\frac{1}{2}be^{2b\varphi}G_{\mu\nu}^2 + \sum_a \left[-\frac{b}{2}e^{-b\varphi}(\partial_{\mu}A_a)^2 - \frac{b}{2}e^{-b\varphi}(\partial_{\mu}B_a)^2 + \frac{b}{2}m_a^2e^{-b\varphi}(A_a^2 + B_a^2) \right] , \quad (3.2)$$

$$\partial_{\nu}(ee^{2b\varphi}G^{\mu\nu}) = -b\epsilon^{\mu\nu\rho}\sum_a(\partial_{\nu}A_a)(\partial_{\rho}B_a) , \quad (3.3)$$

$$\partial_{\mu}(ee^{-b\varphi}g^{\mu\nu}\partial_{\nu}A_a) - \frac{b}{2}\epsilon^{\mu\nu\rho}G_{\mu\nu}\partial_{\rho}B_a + m_a^2ee^{-b\varphi}A_a = 0 , \quad (3.4)$$

$$\partial_{\mu}(ee^{-b\varphi}g^{\mu\nu}\partial_{\nu}B_a) + \frac{b}{2}\epsilon^{\mu\nu\rho}G_{\mu\nu}\partial_{\rho}A_a + m_a^2ee^{-b\varphi}B_a = 0 . \quad (3.5)$$

Our ansatz for the dreibein is

$$(e_{\mu}{}^m) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\sigma} & 0 \\ 0 & 0 & e^{\sigma} \end{pmatrix} , \quad (3.6)$$

for the coordinates $(x^0, x^1, x^2) = (t, x, y)$. If we have the solution

$$\sigma = a \ln r , \quad r \equiv \sqrt{x^2 + y^2} \quad (a : \text{const.}) , \quad (3.7)$$

for a non-integer a , we have a conical singularity [2]. For simplicity we also require the backgrounds

$$\varphi = 0 , \quad G_{\mu\nu} = 0 , \quad B_1 = 0 , \quad A_a = B_a = 0 \quad (a \geq 2) . \quad (3.8)$$

Accordingly, eqs. (3.4) for $a \geq 2$ and (3.5) for $a \geq 1$ are trivially satisfied. From now on, we use $M \equiv m_1$ distinguished from other masses.

We next solve eq. (3.4) for $a = 1$ under the ansatz (3.8):

$$\partial_x^2 A_1 + \partial_y^2 A_1 - M^2 e^{2\sigma} A_1 = A_1''(r) + \frac{1}{r}A_1'(r) - M^2 r^{2a} A_1(r) = 0 , \quad (3.9)$$

where the primes are derivatives by the variables in the parentheses. A new variable

$$\zeta \equiv \frac{M}{a+1}r^{a+1} , \quad r = \left(\frac{a+1}{M}\right)^{1/(a+1)}\zeta^{1/(a+1)} , \quad (3.10)$$

simplifies (3.9) as [8]

$$A_1''(\zeta) + \frac{1}{\zeta}A_1'(\zeta) - A_1(\zeta) = 0 , \quad (3.11)$$

which can be easily solved by

$$A_1 = \alpha K_0(\zeta) = \alpha K_0\left(\frac{M}{a+1}r^{a+1}\right) \quad (\alpha : \text{const.}) , \quad (3.12)$$

as long as

$$a > -1 \quad . \quad (3.13)$$

Here $K_0(z)$ is a “modified Bessel function” related to the Hankel functions (Bessel functions of the third kind) [8] as

$$K_\nu(z) \equiv \frac{\pi i}{2} e^{\nu\pi i/2} H_n^{(1)}(iz) \quad . \quad (3.14)$$

Eq. (3.13) is needed for the damping behaviour at $r \rightarrow \infty$ [8], as is seen from the asymptotic form

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} (1 + \mathcal{O}(z^{-1})) \quad (z \approx \infty) \quad . \quad (3.15)$$

We next study the dilaton field equation (3.2) for the solution (3.12). Now due to $\varphi = 0$,

$$r^{-2a}(\partial_x A_1)^2 + r^{-2a}(\partial_y A_1)^2 + M^2 A_1^2 = r^{-2a} \alpha^2 (\partial_r K_0(\zeta))^2 + M^2 (K_0(\zeta))^2 = 0 \quad (3.16)$$

should vanish. In fact, each term here vanishes desirably, when we take the limit

$$M \rightarrow \infty \quad , \quad (3.17)$$

because of the relations from (3.15) such as

$$\begin{aligned} r^{-a} \partial_r K_0(\zeta) &\approx -M K_0(\zeta) \\ &\approx -M \sqrt{\frac{\pi(a+1)}{2Mr^{a+1}}} \exp\left(-\frac{M}{a+1} r^{a+1}\right) [1 + \mathcal{O}(M^{-1})] \rightarrow 0 \quad (M \rightarrow \infty) \quad . \end{aligned} \quad (3.18)$$

The G -field eq. (3.3) vanishes, due to the vanishing B -fields. The remaining field equation is (3.1), which is re-written in terms of the polar coordinates (r, θ) as

$$R_{rr} = 2 \left(\partial_r^2 \sigma + \frac{1}{r} \partial_r \sigma \right) = 2A_1''(r) + 4M^2 r^{2a} (A_1(r))^2 \quad , \quad (3.19)$$

$$R_{\theta\theta} = r^2 R_{rr} \quad . \quad (3.20)$$

Recall (3.7) and the property of the Green's function [9] that

$$\partial_r^2 (\ln r) + \frac{1}{r} \partial_r (\ln r) = -\frac{\delta(r)}{r} \quad . \quad (3.21)$$

Using these in the middle side of (3.19) implies that

$$-2a \frac{\delta(r)}{r} = 2A_1''(r) + 4M^2 r^{2a} (A_1(r))^2 \quad (3.22)$$

is to be satisfied. In fact, the r.h.s. of (3.22) vanishes everywhere in $r > 0$ under $M \rightarrow \infty$, and the singularity at $r = 0$ can be estimated by the integration

$$\begin{aligned} &\int_0^\infty dr r [2A_1''(r) + 4M^2 r^{2a} (A_1(r))^2] \\ &= 2\alpha^2 (a+1)^2 \int_0^\infty d\zeta \zeta [(K_0'(\zeta))^2 + 2(K_0(\zeta))^2] = 2\alpha^2 (a+1)^2 (c_1 + 2c_0) \quad , \end{aligned} \quad (3.23)$$

where we have used the relation $K'_1(\zeta) = -K_0(\zeta)$, and $c's$ are positive constants:

$$c_0 \equiv \int_0^\infty d\zeta \zeta (K_0(\zeta))^2, \quad c_1 \equiv \int_0^\infty d\zeta \zeta (K_1(\zeta))^2. \quad (3.24)$$

Comparing (3.23) with the l.h.s. of (3.22) yields the condition

$$\alpha = \pm \frac{1}{|a+1|} \sqrt{\frac{-a}{2c_0+c_1}} \quad (-1 < a < 0). \quad (3.25)$$

Note that $a < 0$ is to be satisfied for this to make sense. Eventually, we have established that

$$R_{rr} = -2a \frac{\delta(r)}{r}, \quad (3.26)$$

which in turn implies the satisfaction of (3.20) due to $r\delta(r) \equiv 0$.

We finally inspect whether the Killing spinor equations (2.4b) = (2.4d) = (2.5c) = 0 hold. We start with (2.5c) simplified as

$$[(\sigma_1 \epsilon^i) \cos \theta + (\sigma_2 \epsilon^i) \sin \theta] (r^{-a} \partial_r K_0(\zeta)) + M \epsilon^i K_0(\zeta) \neq 0, \quad (3.27)$$

for the representation $\gamma^{(1)} = i\sigma_1$, $\gamma^{(2)} = i\sigma_2$, $\gamma^{(0)} = -\sigma_3$. We already know that each of these terms vanishes by (3.18) when $M \rightarrow \infty$.⁶ Eq. (2.4d) $\neq 0$ is trivial due to the vanishing φ and G , while the only remaining one is (2.4b) $\neq 0$:

$$D_m(e) \epsilon^i \neq 0, \quad (3.28)$$

which is sufficient for the vanishing commutator

$$[D_{(1)}(e), D_{(2)}(e)] \epsilon^i = -\frac{i}{4} (\sigma_3 \epsilon^i) (g^{11} R_{11} + g^{22} R_{22}) = \frac{i}{2} a (\sigma_3 \epsilon^i) r^{-2a-1} \delta(r) \neq 0. \quad (3.29)$$

The Killing spinor equation (3.28) has a local solution, when (3.29) vanishes under the condition $-2a - 1 > 0$, *i.e.*, $a < -1/2$ consistently with (3.13) and (3.25), so eventually

$$-1 < a < -\frac{1}{2}. \quad (3.30)$$

Since no integer is allowed for a , our mechanism automatically generates the conical singularity [2]. Note also that the case $a = -1$ would correspond to *no* conical singularity.

The satisfaction of Killing spinor equations above, however, is formal in the sense that there is actually no covariantly constant spinors that satisfy the global boundary condition in a conical space-time [1]. This can be most easily seen by solving eq. (3.28) = 0: Using $\omega_1^{(1)(2)} = -ay/r^2$, $\omega_2^{(1)(2)} = ax/r^2$, we get

$$\partial_x \epsilon^i - \frac{ia y}{2r^2} \sigma_3 \epsilon^i = 0, \quad \partial_y \epsilon^i + \frac{ia x}{2r^2} \sigma_3 \epsilon^i = 0, \quad (3.31)$$

⁶We can also make sure the absence of the δ -function singularity at $r = 0$ by an r -integration.

which has only the r -independent solution

$$\epsilon^i(\theta) = \begin{pmatrix} e^{-ia\theta/2} & 0 \\ 0 & e^{+ia\theta/2} \end{pmatrix} \epsilon^i(0) \quad . \quad (3.32)$$

However, this solution does *not* satisfy the usual periodic boundary condition for a fermion: $\epsilon(4\pi) \neq \epsilon(0)$, and therefore the such a covariant spinor does not globally exist. Relevantly, the $N = 2$ supersymmetry is broken down to $N = 0$, and the degeneracy between fermions and bosons will be shifted [1][5], because any θ -dependent fermionic solution violates this boundary condition. In ref. [3] there was a minimal coupling of an $U(1)$ gauge field A_μ in a vector multiplet coupled to the gravitino which compensated the phase discrepancy. In our system a minimal coupling of B_μ induces a dilaton-dependent *negative* definite potential which becomes zero at $|\varphi| = \infty$ with otherwise anti-de Sitter space-time destabilizing the system.⁷ On the other hand, even though the minimal coupling of a vector field A_μ (instead of B_μ) in an independent vector multiplet may have most likely a positive definite potential, we doubt its validity due to inconsistency between the $\psi\partial\varphi$ and ψF -sectors after the \tilde{g} -dependent variations of candidate lagrangian terms,⁸ as simple computations reveal.

To conclude, there is no surviving supersymmetry on this $D = 3, N = 2$ background, because the global periodic boundary condition for the supersymmetry parameter is violated, even though the Killing spinor equations are locally satisfied. However, the cosmological constant still vanishes, since the metric is asymptotically Minkowskian.

4. Conical Singularity in $D = 4$

Once we have understood the mechanism of generating a conical singularity in $D = 3$, our next natural step is its application to $D = 4, N = 1$ supergravity, which is of much more crucial interest for phenomenological model building.

There are two options for the $D = 4, N = 1$ supergravity, either with or without gauging the global axial $U(1)_A$ -symmetry. In this paper we consider the latter, because the $U(1)_A$ -gauge field will not play any important role for our purpose.

Our field content is the supergravity multiplet $D = 4, N = 1$ supergravity (e_μ^m, ψ_μ) coupled to n copies of chiral multiplets (ϕ^i, χ^i) ($i, j, \dots = 1, 2, \dots, n$). In this section we follow the notation of ref. [11] based on the Euclidean signature $(+, +, +, +)$ in ref. [12], except for the local Lorentz indices $a, b, \dots = (1), (2), (3), (4)$. The total lagrangian we need is the sum of lagrangians (32) + (42) in ref. [11] in the case of trivial target space

⁷This situation is usual for automorphism group gauging of supergravities with the gauge field *inside* the supergravity multiplet, such as the gauged $D = 4, N = 8$ supergravity [10], *etc.*

⁸Here ψ is the gravitino field and \tilde{g} is the minimal coupling constant.

dictated by eqs. (47) - (50) therein. The purely bosonic part of the total lagrangian is [11]:

$$e^{-1}\mathcal{L}_B = -\frac{1}{2}R - g^{\mu\nu}(\partial_\mu\phi^i)(\partial_\nu\phi^{*i}) + e^K [3|W|^2 - (D_i W)(D_i^* W^*)] \quad , \quad (4.1)$$

where g is the $U(1)_A$ coupling constant, and

$$D_i W \equiv W_{,i} + K_{,i} W \equiv \frac{\partial W}{\partial \phi^i} + \frac{\partial K}{\partial \phi^i} W \quad . \quad (4.2)$$

As in the $D = 3$ case, we fix the Kähler and holomorphic potentials as

$$K \equiv \sum_{i=1}^n \phi^{*i} \phi^i \quad , \quad W = +\frac{1}{2} \sum_{i=1}^n m_i (\phi^i)^2 = +\frac{1}{2} M (\phi^1)^2 + \frac{1}{2} \sum_{j=2}^n m_j (\phi^j)^2 \quad , \quad (4.3)$$

where as before we take $M \rightarrow \infty$ limit at the end. It turns out that there is no need to add a constant to W to fine-tune the cosmological constant to be zero.⁹ In fact, in terms of $\phi^i \equiv (A_i + iB_i)/\sqrt{2}$ the bosonic potential is

$$V(A_1, B_1, A_2, B_2, \dots, A_n, B_n) = \exp(\sum_k |\phi^k|^2) \left[|M\phi^1 + \phi^{*1}W|^2 + \sum_2^n |m_j\phi^j + \phi^{*j}W|^2 - 3|W|^2 \right] \quad . \quad (4.4)$$

Keeping only the terms up to the cubic, we can easily show that this potential is minimized at its zero value for the v.e.v.'s $\phi^i = 0$. If we set non-zero v.e.v.'s only for A_1 , then

$$V(A_1, 0, 0, 0, \dots, 0, 0) = \frac{1}{2} M^2 A_1^2 e^{A_1^2/2} \left[1 + \frac{1}{8} A_1^2 + \frac{1}{16} A_1^4 \right] = \frac{1}{2} M^2 A_1^2 + \mathcal{O}(A_1^3) \quad . \quad (4.5)$$

We can now get all the relevant bosonic field equations

$$R_{\mu\nu} = -(\partial_\mu A_1)(\partial_\nu A_1) - (\partial_\mu B_1)(\partial_\nu B_1) \quad , \quad (4.6)$$

$$e^{-1} \partial_\mu (e g^{\mu\nu} \partial_\nu A_1) - \frac{\partial}{\partial A_1} V(A_1, 0, 0, 0, \dots, 0, 0) = 0 \quad , \quad (4.7)$$

$$e^{-1} \partial_\mu (e g^{\mu\nu} \partial_\nu B_1) - \frac{\partial}{\partial B_1} V(A_1, B_1, 0, 0, \dots, 0, 0) = 0 \quad , \quad (4.8)$$

where we are considering the case $\phi^j = 0$ ($j = 2, \dots, n$), because V is minimized at these v.e.v.'s, while keeping only A_1 to be non-trivial.

We now fix the ansatz for the vielbein for $(x^\mu) = (x, y, z, t)$ as

$$(e_\mu^a) = \begin{pmatrix} e^\sigma & 0 & 0 & 0 \\ 0 & e^\sigma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \sigma = a \ln \rho \quad , \quad \rho \equiv \sqrt{x^2 + y^2} \quad , \quad (4.9)$$

⁹In this section we do not address ourselves to the problem with the fine-tuning. (Cf. section 5)

like the $D = 3$ case. As for A_1 , we use exactly the same solution as $D = 3$:

$$A_1(\rho) = \alpha K_0(\zeta) = \alpha K_0\left(\frac{M}{a+1}\rho^{a+1}\right) \quad , \quad \zeta \equiv M(a+1)^{-1}\rho^{a+1} \quad . \quad (4.10)$$

We easily see that eq. (4.8) is satisfied, because the first term vanishes for $B_1 = 0$, and we also know that the potential is minimized at $B_1 = 0$. Now eq. (4.7) is re-written as

$$\rho^{-2a} \left(\partial_\rho A_1 + \frac{1}{\rho} \partial_\rho A_1 \right) - M^2 A_1 + \mathcal{O}(A_1^2) \neq 0 \quad , \quad (4.11)$$

by (4.5). This is satisfied as (3.16), because $\mathcal{O}(A_1^2)$ damps much faster than other terms as $M \rightarrow \infty$.

The remaining field equation to be examined is (4.6). All the terms are of the same pattern as the $D = 3$ case, and the rest boils down to an analog of (3.25) *except for* $2c_0$ in (3.25) replaced by c_0 now:

$$\begin{aligned} \int_0^\infty d\rho \rho R_{\rho\rho} &= 2 \int_0^\infty d\rho \rho \left(\partial_\rho^2 \sigma + \frac{1}{\rho} \partial_\rho \sigma \right) \\ &= \int_0^\infty d\rho \left[(\partial_\rho A_1)^2 + M^2 \rho^{2a} A_1^2 \right] = \alpha^2 (a+1)^2 (c_0 + c_1) \quad , \end{aligned} \quad (4.12)$$

$$\alpha = \pm \frac{1}{|a+1|} \sqrt{\frac{-a}{c_0+c_1}} \quad (-1 < a < 0) \quad . \quad (4.13)$$

This concludes the satisfaction of all the bosonic field equations after taking the limit $M \rightarrow \infty$, and the resemblance between the $D = 4$ and $D = 3$ cases is clear.

We finally analyze the Killing spinor equations by the supertranslations of fermions [11]:

$$\delta_Q \psi_\mu = 2\mathcal{D}_\mu \epsilon + \frac{1}{2} (K_{,i} \partial_\mu \phi^i - K_{,i^*} \partial_\mu \phi^{*i}) \gamma_5 \epsilon + \frac{1}{2} e^{K/2} W \gamma_\mu \epsilon \quad , \quad (4.14)$$

$$\delta_Q \chi^i = \sqrt{2} \gamma^\mu \epsilon \partial_\mu \phi^{*i} - \sqrt{2} e^{K/2} \epsilon D^i W \quad . \quad (4.15)$$

These equations resemble the $D = 3$ case. Eq. (4.15) = 0 has the linear terms similar to (3.27), and higher order terms, both vanishing at $M \rightarrow \infty$.

The non-trivial Killing equation is (4.14) = 0, which has a new term of W compared with (3.31). However, this term is at least bilinear in A_1 which is easily shown to vanish under $M \rightarrow \infty$. In fact, (4.14) = 0 is solved by

$$\epsilon(\theta) = I_2 \otimes \begin{pmatrix} e^{-ia\theta/2} & 0 \\ 0 & e^{+ia\theta/2} \end{pmatrix} \epsilon(0) \quad , \quad (4.16)$$

for $\theta \equiv \arctan(y/x)$. Corresponding to (3.29) we have the condition

$$\begin{aligned} [D_{(1)}(e), D_{(2)}(e)] \epsilon &= -\frac{i}{4} (I_2 \otimes \sigma_3) \epsilon (g^{11} R_{11} + g^{22} R_{22}) \\ &= \frac{i}{2} a (I_2 \otimes \sigma_3) \epsilon \rho^{-2a-1} \delta(\rho) \neq 0 \quad , \end{aligned} \quad (4.17)$$

yielding

$$-1 < a < -\frac{1}{2} . \quad (4.18)$$

The representation we have used is $\gamma^{(4)} = -\sigma_1 \otimes I_2$, $\gamma^{(i)} = +\sigma_2 \otimes \sigma_i$ ($i = 1, 2, 3$), $\gamma_5 = +\sigma_3 \otimes I_2$. Unlike that of $D = 3$ in ref. [3], we have no $U(1)_A$ coupling to ϵ , so that we have no cancellation between the Lorentz connection and A_μ .¹⁰ Therefore as our previous $D = 3$ case, the covariantly constant spinor can exist only locally but not globally, violating the boundary condition for a fermion: $\epsilon(4\pi) \neq \epsilon(0)$. Accordingly, the original degeneracy between fermions and bosons under supersymmetry such as the masses or couplings will be lifted [1][5], which may be useful for phenomenological applications [13]. However, the vanishing of the cosmological constant is not disturbed by this supersymmetry breaking, owing to the local satisfaction of bosonic field equations, and the metric is asymptotically Euclidean.

5. Duality between $D = 4$ and $D = 3$

We mention briefly the strong coupling duality [5] between the two systems in $D = 4$ and $D = 3$, that may explain the automatic zero-ness of the cosmological constant with broken supersymmetry in the former.

As eq. (2.7) for $a = b$ indicates, the field redefinitions

$$\begin{aligned} (\tilde{e}_\mu{}^m, \tilde{\psi}_\mu, \tilde{B}_\mu, \tilde{\chi}) &\equiv (e^{-b\varphi} e_\mu{}^m, e^{-b\varphi/2} \psi_\mu, B_\mu, e^{+b\varphi/2} \chi) , \\ (\tilde{A}_a, \tilde{B}_a, \tilde{\lambda}_a) &\equiv (e^{-b\varphi/2} A_a, e^{-b\varphi/2} B_a, e^{+b\varphi/2} \lambda_a) , \end{aligned} \quad (5.1)$$

make the dilaton-dependence manifest as the string coupling constant [5]:

$$\mathcal{L}_{\text{SG}} = e^{b\varphi} \tilde{\mathcal{L}}_{\text{SG}} , \quad \mathcal{L}_{\text{SM}} = e^{b\varphi} \tilde{\mathcal{L}}_{\text{SM}} , \quad \mathcal{L}_m = e^{b\varphi} \tilde{\mathcal{L}}_m \Big|_{m_a \rightarrow e^{-b\varphi} \tilde{m}_a} , \quad (5.2)$$

where the *tilded* lagrangians have only the *tilded* fields. In $\mathcal{L}_{\text{SG}} + \mathcal{L}_{\text{SM}}$, the dilaton without derivative appears only in the over-all factor $e^{b\varphi}$. In \mathcal{L}_m , the old masses m_a are also formally replaced by $e^{-b\varphi} \tilde{m}_a$. From $\mathcal{L}_{\text{SG}} + \mathcal{L}_{\text{SM}}$ the string coupling constant λ is identified as $\lambda^{-2} = e^{b\varphi}$ [5].

We can relate this result to the $D = 4$, $N = 1$ supergravity plus chiral multiplet by the dimensional reduction on $\mathbb{R}^3 \otimes S^1$ with a circle S^1 of radius $r = e^{-b\varphi}$ as¹¹

$$d\hat{s}^2 = \hat{g}_{\hat{\mu}\hat{\nu}} d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} = e^{2b\varphi} \tilde{g}_{\mu\nu} dx^\mu dx^\nu - e^{-2b\varphi} (d\hat{x}^3)^2 , \quad (5.3)$$

¹⁰Even if we introduce an $U(1)_A$ minimal couplings, there will be no cancellation due to the difference in the γ -matrix structure in $D = 4$.

¹¹Here we switch to the signature $(+, -, -, -)$.

where all the *hatted* quantities are for $D = 4$, while the *tilded* $\tilde{g}_{\mu\nu}$ accords with (5.1). Accordingly, the $D = 4$, $N = 1$ supergravity multiplet $(\hat{e}_{\hat{\mu}}^{\hat{m}}, \hat{\psi}_{\hat{\mu}})$ is reduced to the $D = 3$, $N = 2$ supergravity $(e_{\mu}^m, \psi_{\mu}^i, B_{\mu}, \chi^i, \varphi)$, while the chiral multiplet $(\hat{\phi}_i, \hat{\chi}_i)$ is reduced to the scalar multiplet (A_a, B_a, λ_a^i) .¹² Now from the $D = 4$, $N = 1$ lagrangian [11] of section 4, we can identify the radius [5]

$$r = e^{-b\varphi} = \lambda^2 \quad , \quad (5.4)$$

and its bosonic part¹³ after this dimensional reduction agrees with that of $\mathcal{L}_{\text{SG}} + \mathcal{L}_{\text{SM}}$ in (5.2) constructed within $D = 3$.

This relationship satisfies the criterion of consistency for duality in ref. [5], namely if the strong coupling limit $\lambda \rightarrow \infty$ in $D = 3$, then the S^1 radius $r \rightarrow \infty$, implying the promotion (or “oxidation”) of the $D = 3$ system to a $D = 4$ system with the Poincaré invariance. Since $e^{b\varphi} m_a = m_a / \lambda^2$, our limiting procedure $m_1 \equiv M \rightarrow \infty$ may well be interpreted as the effective mass $\tilde{M} \equiv M / \lambda^2$ kept finite in $D = 4$. Now the question of the fine-tuning of the $D = 4$ cosmological constant is solved, due to the fundamental $D = 3$ theory automatically yielding the zero cosmological constant under this duality, absorbing the undesirable massless dilaton at the same time [5].

6. Concluding Remarks

In this paper we have presented a mechanism of generating a conical singularity in $D = 3$ directly applicable to $D = 4$. As a by-product, $D = 3$, $N = 2$ supergravity lagrangians are presented in terms of the supergravity multiplet with a dilaton and an antisymmetric tensor coupled to massive scalar multiplets, which have not been presented elsewhere to our knowledge.

We have also seen that the strong coupling duality relation outlined in [5] is realized between the systems of two dimensionalities, namely when the string coupling constant λ in $D = 3$ is taken to infinity, the S^1 radius for the compactification from $D = 4$ grows. This implies that the $\lambda \rightarrow \infty$ limit in $D = 3$ is equivalent to a $D = 4$ system with the Minkowskian metric and Poincaré invariance. To put it differently, this duality can be regarded as dimensional “oxidation” from $D = 3$ to $D = 4$ which is the reversed process of the usual dimensional “reduction” from $D = 4$ to $D = 3$. Accordingly, the supersymmetry is broken in $D = 4$ with a conical singularity, but the zero-ness of the cosmological constant is automatically realized without fine-tuning by this duality. Our peculiar limit $M \rightarrow \infty$ we adopted in our mechanism maintains the finite effective mass in the resulting $D = 4$.

The limit $M \rightarrow \infty$ is also natural from another viewpoint. For example, the point-mass source in ref. [2] creating conical singularities should correspond to the point particle

¹²The index-conventions follow the corresponding section of each multiplet.

¹³Due to supersymmetry, the same is also true for the fermionic part.

limit in a field theory, and such a limit must be realized by taking the mass of a particle to infinity in order to enhance the “point particle” effect, suppressing the “wave” effect. The $M \rightarrow \infty$ may well be related to the familiar limit $M_{\text{Pl}} \equiv 1/\kappa \rightarrow \infty$ [13] for the “low energy” physics such as local supersymmetric grand unifications below the Planck mass M_{Pl} . Note also that important relations in our procedure such as the condition (3.30) or (4.18) do not depend on the value of M , so that the $M \rightarrow \infty$ limit makes sense. This non-trivial feature indicates that our $M \rightarrow \infty$ limit has some fundamental significance controlling the conical singularity of the space-time in a “topological” way. It is interesting that our mechanism has a finite breaking effect by $a \approx \mathcal{O}(1)$ instead of $M/M_{\text{Pl}} \rightarrow 0$ [3] well below M_{Pl} .

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